<u>Final Exam</u>:

Topics: Filtering up to Energy Minimization,

NOT including Segmentation

i.e. the Topics in Final Practice.

Since DFT is highly related to the topics,

You should also do some revision on DFT.

$f = (f_{ij})_{0 \le i,j \le N-1} \in \mathbb{R}^{N \times N}$,

$$G_x(f)(x,y) = \frac{1}{4} f(x+1, y) + \frac{1}{2} f(x,y) + \frac{1}{4} f(x-1, y)$$

$$G_y(f)(x,y) = \frac{1}{4} f(x, y+1) + \frac{1}{2} f(x,y) + \frac{1}{4} f(x, y-1)$$

G_x, G_y linear and Shift / Position invariant

$\Rightarrow$ both are convolution.

$$G_x \circ G_y(f) = G_x(G_y(f))$$

$G_x \circ G_y(f)(x,y)$

$$= \frac{1}{4} G_y(f)(x+1, y) + \frac{1}{2} G_y(f)(x,y) + \frac{1}{4} G_y(f)(x-1, y)$$

$$= \frac{1}{4}\left(\frac{1}{4} f(x+1, y+1) + \frac{1}{2} f(x+1, y) + \frac{1}{4} f(x+1, y-1)\right)$$

$$+ \frac{1}{2}\left(\frac{1}{4} f(x, y+1) + \frac{1}{2} f(x,y) + \frac{1}{4} f(x, y-1)\right)$$

$$+ \frac{1}{4}\left(\frac{1}{4} f(x-1, y+1) + \frac{1}{2} f(x-1, y) + \frac{1}{4} f(x-1, y-1)\right)$$

$$= \frac{1}{16} f(x-1, y-1) + \frac{1}{8} f(x-1, y) + \frac{1}{16} f(x-1, y+1)$$

$$+ \frac{1}{8} f(x, y-1) + \frac{1}{4} f(x,y) + \frac{1}{8} f(x, y+1)$$

$$+ \frac{1}{16} f(x+1, y-1) + \frac{1}{8} f(x+1, y) + \frac{1}{16} f(x+1, y+1)$$

$$\therefore \quad G_x \circ G_y(f) = h * f$$

$$= \begin{bmatrix} 1/4 & 1/8 & 0 & \cdots & 0 & 1/8 \\ 1/8 & 1/16 & 0 & \cdots & 0 & 1/16 \\ 0 & 0 & & & & \vdots \\ \vdots & \vdots & & & & \vdots \\ 1/8 & 1/16 & 0 & \cdots & 0 & 1/16 \end{bmatrix} * f.$$

In General, convolution is associative.

i.e. $h_1 * (h_2 * f) = (h_1 * h_2) * f$

Note: $$G_x(f) = \begin{bmatrix} 1/2 & 0 & \cdots & 0 \\ 1/4 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & & & \\ 1/4 & 0 & \cdots & 0 \end{bmatrix} * f$$

$$G_y(f) = \begin{bmatrix} 1/2 & 1/4 & 0 & \cdots & 0 & 1/4 \\ 0 & 0 & & \cdots & & 0 \\ \vdots & \vdots & & \ddots & & \\ 0 & 0 & & & & 0 \end{bmatrix} * f$$

$$G_x \circ G_y(f) = h_1 * (h_2 * f)$$
$$= (h_1 * h_2) * f$$

$$h_1 * h_2 = \begin{bmatrix} 1/4 & 1/8 & 0 & \cdots & 0 & 1/8 \\ 1/8 & 1/16 & 0 & \cdots & 0 & 1/16 \\ 0 & 0 & & & & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & & & & 0 \\ 1/8 & 1/16 & 0 & \cdots & 0 & 1/16 \end{bmatrix}$$

Suppose $f, g = 0$ on ∂D

Consider the Energy:

$$E(f) = \int_D (f(x,y) - g(x,y))^2 + \int |\nabla f|^2$$

Let f minimizer of E, $v = 0$ on D,

$$S(\varepsilon) = E(f + \varepsilon v)$$

$$\frac{d}{d\varepsilon} S(\varepsilon) = \frac{d}{d\varepsilon} \int_D (f - g + \varepsilon v)^2$$

$$+ \frac{d}{d\varepsilon} \int_D \langle \nabla f + \varepsilon \nabla v, \nabla f + \varepsilon \nabla v \rangle$$

$$= \frac{d}{d\varepsilon} \int_D (f - g + \varepsilon v)^2$$

$$+ \frac{d}{d\varepsilon} \int_D |\nabla f|^2 + 2\varepsilon \nabla f \cdot \nabla v + \varepsilon^2 |\nabla v|^2$$

$$= 2 \int_D (f - g + \varepsilon v) v$$

$$+ 2 \int_D \nabla f \cdot \nabla v + 2\varepsilon |\nabla v|^2$$

f minimizer $\Rightarrow \frac{d}{d\varepsilon} S(\varepsilon) \big|_{\varepsilon = 0} = 0$

$$\Rightarrow 0 = \int_D (f - g) v + \int_D \nabla f \cdot \nabla v$$

By Divergence Thm (Integration by Parts),

$$\int_D \nabla f \cdot \nabla v = \int_{\partial D} v(\nabla f \cdot \vec{n}) - \int_D v \Delta f$$

By assumption, $v \equiv 0$ on ∂D,

$$\int_D \nabla f \cdot \nabla v = -\int_D v \Delta f$$

Then,

$$0 = \int_D (f - g - \Delta f) v$$

v is arbitrary,

$$\Rightarrow \begin{cases} 0 = f - g - \Delta f \\ f = g = 0 \quad \text{on } \partial D \end{cases}$$

To do:

Solve $\begin{cases} 0 = f \cdot g - \Delta f \\ f = g = 0 \quad \text{on } \partial D \end{cases}$ numerically.

Now, f, g are images $\in \mathbb{R}^{N \times N}$

$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ is continuous differential operator,

we have to discretized it such that it can be applied to matrix.

Taylor Expansion:

$$(1) - f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \dots$$
$$(2) - f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2}h^2 + \dots$$

$(1) + (2)$:

$$f(x+h) + f(x-h) = 2f(x) + f''(x)h^2 + \text{higher order terms.}$$

$$\Rightarrow \quad f''(x) \approx \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}$$

in matrix, we set $h = 1$, approximate $\frac{\partial^2 f}{\partial x^2}$ by

$$\frac{\partial^2 f}{\partial x^2}(x,y) \approx f(x+1,y) + f(x-1,y) - 2f(x,y)$$

$$\therefore \Delta f(x,y) \approx -4f(x,y) + f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1)$$

So we write the PDE as:

find $f \in \mathbb{R}^{N \times N}$ s.t.

$$0 = f(x,y) - g(x,y) - [-4f(x,y) + f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1)]$$

$$\Leftrightarrow \quad g(x,y) = 5f(x,y) - f(x+1,y) - f(x-1,y) - f(x,y+1) - f(x,y-1) \qquad \text{— } (*)$$

Write $\vec{g}$ be the column vector form of g,

$\vec{f}$ be the column vecte form of f,

write RHS as $O(\vec{f})$

Note O is linear:

$$O(\alpha \vec{f}_1 + \beta \vec{f}_2) = \alpha O(\vec{f}_1) + \beta O(\vec{f}_2)$$

$\therefore$ O is some linear transform

and we can write (*)

as $\vec{g} = A\vec{f}$, $A \in \mathbb{R}^{N^2 \times N^2}$ some matrix

$$f = \begin{bmatrix} | & | & & | \\ \vec{f}_0 & \vec{f}_1 & \cdots & \vec{f}_{N-1} \\ | & | & & | \end{bmatrix}, \quad \vec{f} = \begin{bmatrix} \vec{f}_0 \\ \vec{f}_1 \\ \vdots \\ \vec{f}_{N-1} \end{bmatrix}$$

$0 \leq j \leq N^2 - 1$,

Write $j = m_j N + n_j$, $m_j = \left\lfloor \frac{j}{N} \right\rfloor$, $n_j = j \bmod N$

Note $0 \leq m_j, n_j \leq N-1$

Then

$$\vec{f}(j) = f(m_j, n_j)$$

$$(A\vec{f})_i = \sum_{j=0}^{N^2-1} A(i,j)\,\vec{f}(j)$$

$$\iff G(f)(m_i,n_i) = \sum_{j=0}^{N^2-1} A(i,j)\, f(m_j,n_j)$$

$$\iff 5f(m_i,n_i) - f(m_i+1,n_i) - f(m_i-1,n_i) - f(m_i,n_i+1) - f(m_i,n_i-1) = \sum_{j=0}^{N^2-1} A(i,j)\, f(m_j,n_j)$$

By periodicity,

$$= 5f(m_i,n_i) - f(m_i^+,n_i) - f(m_i^-,n_i) - f(m_i,n_i^+) - f(m_i,n_i^-)$$

where $m_i^+ = \begin{cases} 0, & \text{if } m_i+1 = 0 \\ m_i+1, & \text{else} \end{cases}$, $m_i^- = \begin{cases} N-1, & \text{if } m_i-1 = -1 \\ m_i-1, & \text{else} \end{cases}$

$n_i^+ = \begin{cases} 0, & \text{if } n_i+1 = 0 \\ n_i+1, & \text{else} \end{cases}$, $n_i^- = \begin{cases} N-1, & \text{if } n_i-1 = -1 \\ n_i-1, & \text{else} \end{cases}$

$$\begin{aligned} = {} & A(i,i)\, f(m_i,n_i) \\ & + A(i^+,i)\, f(m_i^+,n_j) \\ & + A(i^-,i)\, f(m_i^-,n_j) \\ & + A(i,i_+)\, f(m_i,n_i^+) \\ & + A(i,i_-)\, f(m_i,n_i^-) \\ & + \text{the rest.} \end{aligned}$$

where $i^+ = m_i^+ N + n_j$

$i^- = m_i^- N + n_j$

$i_+ = m_i N + n_i^+$

$i_- = m_i N + n_i^-$

$$\Rightarrow \quad \begin{aligned} A(i,i) &= 5, \\ A(i_+, i) &= -1, \\ A(i_-, i) &= -1, \\ A(i, i_+) &= -1, \\ A(i, i_-) &= -1, \quad \forall\, 0 \le i \le N^2 - 1 \end{aligned}$$

the rest are zero.